

Structural stability of pairs of matrices under contragredient equivalence

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Abstract

A complex matrix pencil $A - \lambda B$ is called structurally stable if there exists its neighborhood in which all pencils are strictly equivalent to this pencil. We describe all complex matrix pencils that are structurally stable. It is shown that there are no pairs (M, N) of $m \times n$ and $n \times m$ complex matrices ($m, n \geq 1$) that are structurally stable under the contragredient equivalence $(S^{-1}MR, R^{-1}NS)$, in which S and R are nonsingular.

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1. Introduction

The concept of structural stability, first introduced by A. A. Andronov and L. S. Pontryagin in 1937 in the qualitative theory of dynamical systems, in the sense of structurally stable elements being those whose behavior does not change when applying small perturbation [1], has been largely studied by many authors in different contexts for example, in the scenario of Linear Algebra we can found [4], [5], [7], [8], among others.

In this work, we consider pairs of complex matrices (A, B) , in which $A \in M_{n \times m}(\mathbb{C})$ and $B \in M_{m \times n}(\mathbb{C})$, and the contragredient equivalence relation, which generalizes of four basic equivalence relations: similarity, consimilarity, complex orthogonal equivalence, and unitary equivalence. We can reduce (A, B) by *transformations of contragredient equivalence*

$$(A, B) \mapsto (SAR^{-1}, RBS^{-1}), \quad S \text{ and } R \text{ are nonsingular.} \quad (1)$$

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The canonical form of (A, B) with respect to these transformations was obtained by Dobrovol'skaya and Ponomarev [6] and, independently, by Horn and Merino[10]:

each pair (A, B) is contragrediently equivalent to a direct sum, uniquely determined up to permutation of summands, of pairs of the types $(I_r, J_r(\lambda))$, $(J_r(0), I_r)$, (F_r, G_r) , (G_r, F_r) , (2)

in which

$$J_r(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix} \quad (\lambda \in \mathbb{C}),$$

$$F_r := \begin{bmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & 0 \\ 0 & & 1 \end{bmatrix}, \quad G_r := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}$$

are $r \times r$, $r \times (r-1)$, $(r-1) \times r$ matrices and

$$(A_1, B_1) \oplus (A_2, B_2) := (A_1 \oplus A_2, B_1 \oplus B_2).$$

Note that $(F_1, G_1) = (0_{10}, 0_{10})$; we denote by 0_{mn} the zero matrix of size $m \times n$, where $m, n \in \{0, 1, 2, \dots\}$.

A notion of a miniversal deformation was introduced by Arnold [2, 3]. He constructed a miniversal deformation of a Jordan matrix J ; i.e., a simple normal form to which all matrices $J + E$ close to J can be reduced by similarity transformations that smoothly depend on the entries of E . Garca-Planas and Sergeichuk [9] constructed a miniversal deformation of a canonical pair (2) for contragredient equivalence (8).

A *miniversal deformation* of a pair of matrices (A, B) is a family of pairs of matrices in a neighborhood of the canonical reduced form (A_c, B_c) of the given pair with the minimal number of parameters, to which all pair of matrices (C, D) close to (A_c, B_c) can be reduced by contragredient equivalence transformations that smoothly depend on the entries of (A, B) .

The aim of this paper is to characterize the structural stability of a pair of matrices, with regard to this equivalence relation, in terms of the complete system of invariants given in [10]. A *structurally stable pair* is an pair having a neighborhood contained in its equivalence class that is to say a small perturbation of it gives rise to an pair equivalent to it.

Since the unstability of a pair of matrices is directly related to the existence of a non-zero miniversal deformations, then the basic tool of studying a structural stability is to consider any miniversal deformation of a pair of matrices (for example, the simplest one, obtained in [9]), or any other (see [11]).

The following theorem is the main result of the article.

Theorem 1. *Let (A, B) be a pair of $m \times n$ and $n \times m$ matrices of the form (7). Then (A, B) is stable under the contragredient equivalence if and only if the canonical form (7) of (A, B) has the form*

$$\begin{aligned} & \bigoplus (F_r, G_r) \oplus (F_{r+1}, G_{r+1}) \quad \text{or} \quad \bigoplus (G_r, F_r) \oplus (G_{r+1}, F_{r+1}) \\ & \text{or} \\ & \bigoplus (F_r^T, G_r^T) \oplus (F_{r+1}^T, G_{r+1}^T) \quad \text{or} \quad \bigoplus (G_r^T, F_r^T) \oplus (G_{r+1}^T, F_{r+1}^T) \end{aligned} \tag{3}$$

for some $r \geq 1$.

2. Miniversal deformations

In this section, we recall the miniversal deformations of canonical pairs (2) for contragredient equivalence constructed by García-Planas and Sergeichuk [9]. Let us consider

$$\begin{aligned} \Phi^{\mathbb{F}}(\lambda_i) &:= \text{diag}(J_{s_{i1}}^{\mathbb{F}}(\lambda_i), J_{s_{i2}}^{\mathbb{F}}(\lambda_i), \dots), \\ s_{i1} &\geq s_{i2} \geq \dots \end{aligned}$$

Clearly, every square matrix over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ is similar to a matrix of the form

$$\bigoplus_i \Phi^{\mathbb{F}}(\lambda_i), \quad \lambda_i \neq \lambda_j \text{ if } i \neq j, \tag{4}$$

uniquely determined up to permutations of summands.

Let

$$\mathcal{H} = [H_{ij}] \tag{5}$$

be a parameter block matrix with $p_i \times q_j$ blocks H_{ij} of the form

$$H_{ij} = \begin{bmatrix} * \\ \vdots \\ 0 \\ * \end{bmatrix} \text{ if } p_i \leq q_j, \quad H_{ij} = \begin{bmatrix} 0 \\ * \dots * \end{bmatrix} \text{ if } p_i > q_j, \tag{6}$$

where the stars denote independent parameters.

Let (A, B) be a pair of matrices of the form

$$(I, C) \oplus \bigoplus_{j=1}^{t_1} (I_{r_{1j}}, J_{r_{1j}}) \oplus \bigoplus_{j=1}^{t_2} (J_{r_{2j}}, I_{r_{2j}}) \oplus \bigoplus_{j=1}^{t_3} (F_{r_{3j}}, G_{r_{3j}}) \oplus \bigoplus_{j=1}^{t_4} (G_{r_{4j}}, F_{r_{4j}}) \quad (7)$$

where C is of the form (4) and $r_{i1} \geq r_{i2} \geq \dots \geq r_{it_i}$. One of the simplest miniversal \mathbb{F} -deformations of the canonical pair (7) under contragredient equivalence over $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$ was constructed in [9]. It is the direct sum of (I, \tilde{C}) (\tilde{C} is a simplest miniversal \mathbb{F} -deformation of C under similarity, see Theorem 2.1 in [9]) and

$$\left(\left[\begin{array}{c|c|c} \bigoplus_j I_{r_{1j}} & 0 & 0 \\ \hline 0 & \bigoplus_j J_{r_{2j}} + \mathcal{H} & \mathcal{H} \\ \hline 0 & \mathcal{H} & \begin{smallmatrix} P_3 & \mathcal{H} \\ 0 & Q_4 \end{smallmatrix} \end{array} \right], \left[\begin{array}{c|c|c} \bigoplus_j J_{r_{1j}} + \mathcal{H} & \mathcal{H} & \mathcal{H} \\ \hline \mathcal{H} & \bigoplus_j I_{r_{2j}} & 0 \\ \hline \mathcal{H} & 0 & \begin{smallmatrix} Q_3 & 0 \\ \mathcal{H} & P_4 \end{smallmatrix} \end{array} \right] \right),$$

where

$$P_l = \begin{bmatrix} F_{r_{l1}} + H & H & \cdots & H \\ & F_{r_{l2}} + H & \ddots & \vdots \\ & & \ddots & H \\ 0 & & & F_{r_{lit_l}} + H \end{bmatrix}, \quad Q_l = \begin{bmatrix} G_{r_{l1}} & & & 0 \\ H & G_{r_{l2}} & & \\ \vdots & \ddots & \ddots & \\ H & \cdots & H & G_{r_{lit_l}} \end{bmatrix},$$

($l = 3, 4$), \mathcal{H} and H are matrices of the form (5) and (6), and the stars denote independent parameters.

3. Proof of the main result

Let us prove that if (A, B) contains a pair of the form

- (1) $(I_r, J_r(\lambda)), \quad r \geq 1, \lambda \in \mathbb{R} \cup \{0\},$
- (2) $(J_r(0), I_r),$
- (3) $(F_m^*, G_m^*) \oplus (F_n^\dagger, G_n^\dagger),$ in which $*, \dagger = \{\emptyset, T\}, \quad 1 \leq m \leq n, \quad m \neq n+1, \quad n \neq m+1, \text{ or}$
- (4) $(F_m^*, G_m^\dagger) \oplus (F_n^\dagger, G_n^*),$ in which $*, \dagger = \{\emptyset, T\}$ and $1 \leq m \leq n, \quad m \neq n+1,$

then (A, B) is not stable, i.e., any arbitrary small perturbations does not change the canonical form of the matrix pair.

It is sufficient to consider a case when (A, B) equals to one of the summand of type (1)–(4).

Case 1: $(A, B) = (I_r, J_r(\lambda)), r \geq 1$.

We may assume that $\lambda = 0$ since there exists an arbitrary small deformation of $J_r(\lambda)$ with Jordan canonical form J' if and only if there exists an arbitrary small deformation of $J_r(\lambda) - \lambda I_r$ with JCF $J' - \lambda I$.

Due to (2), the miniversal deformation of $(I_r, J_r(0)), r \geq 1$ has the form

$$\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_r \end{bmatrix} \right).$$

Since we deform only Jordan block it means that each deformation change the canonical form, so (A, B) is unstable.

Case 2: $(A, B) = (J_r(0), I_r), r \geq 1$. Analogously, due to (2), the miniversal deformation of $(J_r(0), I_r), r \geq 1$ has the form

$$\left(\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_r \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \right).$$

And so, (A, B) is unstable.

Case 3.1: $(A, B) = (F_r, G_r) \oplus (F_m^T, G_m^T), 1 \leq m \leq n, m \neq n+1, n \neq m+1$.

$$\left(\left[\begin{array}{ccc|ccc} 1 & & 0 & & & \\ 0 & \ddots & & & 0 & \\ & \ddots & 1 & & & \\ 0 & & 0 & & & \\ \hline & & & 1 & 0 & 0 \\ 0 & & & & \ddots & \ddots \\ & & & 0 & & 1 & 0 \end{array} \right], \left[\begin{array}{ccc|ccc} 0 & 1 & & 0 & & \\ & \ddots & \ddots & & 0 & \\ 0 & & 0 & 1 & & \\ \hline & & & 0 & 0 & 0 \\ & & & & 1 & \ddots \\ & & & & & \ddots & 0 \\ & & & & & & 0 & 1 \end{array} \right] \right).$$

Note that obtained matrix has zero row and column, i.e. it is singular. Due to that fact, we can always by arbitrary small deformation reduce it to the

matrix with the full rank and so obtain a pair with a canonical form which can consist of summands of types (F_r, G_r) and $(J_r(0), I_r)$ (or/and $(I_r, J_r(0))$) or (F_r^T, G_r^T) and $(J_r(0), I_r)$ (or/and $(I_r, J_r(0))$), but not summands of types (F_r, G_r) and (F_r^T, G_r^T) simultaneously, i.e., we get different canonical form and so, unstable pair.

Case 3.2: $(A, B) = (G_r, F_r) \oplus (G_m^T, F_m^T), 1 \leq m \leq n, m \neq n+1, n \neq m+1$. This case can be consider analogously with Case 3.1 by rearranging elements of pair and taking the transposition.

Case 3.3: $(A, B) = (F_m, G_m) \oplus (F_n, G_n), 1 \leq m \leq n, m \neq n+1, n \neq m+1$.

$$\left(\left[\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & & \\ & & \ddots & 1 & & \\ 0 & & & 0 & & \\ \hline & & & 1 & & 0 \\ & & & 0 & \ddots & \\ & 0 & & & \ddots & 1 \\ & & & 0 & & 0 \end{array} \right], \left[\begin{array}{ccc|ccc} 0 & 1 & & 0 & & \\ & \ddots & \ddots & & & 0 \\ & & \ddots & 0 & 1 & \\ 0 & & & 0 & & 0 \\ \hline & & & 0 & 1 & 0 \\ & & & & \ddots & \ddots \\ & & & 0 & & 0 & 1 \end{array} \right] \right). \quad (8)$$

Each nonzero miniversal deformation of (8) has the form

$$\left(\left[\begin{array}{c|c} F_m & 0 \\ \hline 0 & F_n \end{array} \right], \left[\begin{array}{c|c} G_m & 0 \\ \hline 0 & G_n \end{array} \right] \right) \quad (9)$$

in which the last m stars in the first row of the second matrix are zeros.

Case 3.4: $(A, B) = (G_m, F_m) \oplus (G_n, F_n), 1 \leq m \leq n, m \neq n+1, n \neq m+1$. This case can be consider analogously with Case 3.3 by rearranging elements of pair and taking the transposition.

Lemma 1. *Each pair of $n \times (n-1)$ matrices of the form*

$$\left(\left[\begin{array}{cccc} 1 & * & & * \\ 0 & 1 & \ddots & \\ & 0 & \ddots & * \\ & & \ddots & 1 \\ 0 & & & 0 \end{array} \right], \left[\begin{array}{cccc} * & * & & * \\ 1 & * & \ddots & \\ & 1 & \ddots & * \\ & & \ddots & * \\ 0 & & & 1 \end{array} \right] \right) \quad (10)$$

reduces to (F_n, G_n) by simultaneous additions of columns from left to right and simultaneous additions of rows from the bottom to up.

Proof. Consider the subpair \mathcal{P} of (10) obtained by removing the last row and last column in each of the matrices (10). We reduce (10) by simultaneous additions of columns of its matrices from left to right and simultaneous additions of rows from the bottom to up. Reasoning by induction on n , we reduce the subpair \mathcal{P} to (F_{n-1}, G_{n-1}) and obtain (10) in which all stars are zero except for stars of the last columns. We make zero the stars of the last column in the first matrix by adding the other columns simultaneously in both matrices; then we make zero the stars of the last column in the second matrix by adding the last row. \square

Let b_{1k} be the first nonzero star in (9). Divide by b_{1k} the first horizontal strip (consisting of m rows) in the matrices, then multiply by b_{1k} their first vertical strips and obtain a pair

$$\left(\begin{array}{c|c|c|c} \begin{array}{c} 1 \\ 0 \vdots 1 \\ 0 \end{array} & & & \\ \hline & \begin{array}{c} 1 \\ 0 \vdots 1 \\ 0 \end{array} & & \\ \hline & & \begin{array}{c} 1 \\ 0 \vdots 1 \\ 0 \end{array} & \\ \hline & & & \begin{array}{c} 1 \\ 0 \vdots 1 \\ 0 \end{array} \end{array} \right), \quad \left(\begin{array}{c|c|c|c} \overbrace{0 \dots 0}^k 1 & \overbrace{* \dots *}^{l \geq m} & & \\ \hline \begin{array}{c} 0 \\ 1 \vdots 0 \\ 1 \end{array} & & & \\ \hline & \begin{array}{c} 0 \\ 1 \vdots 0 \\ 1 \end{array} & & \\ \hline & & \begin{array}{c} 0 \\ 1 \vdots 0 \\ 1 \end{array} & \\ \hline & & & \begin{array}{c} 0 \\ 1 \vdots 0 \\ 1 \end{array} \end{array} \right) \begin{array}{l} \left. \begin{array}{c} \text{first } m \text{ rows} \\ \text{last } m \text{ rows} \end{array} \right\} m \\ \downarrow (-1) \\ \left. \begin{array}{c} \text{first } m \text{ rows} \\ \text{last } m \text{ rows} \end{array} \right\} m \end{array} \right) \begin{array}{l} \leftarrow (+1) \end{array} \quad (11)$$

of the form (9), in which $b_{1k} = 1$. Make zero the unit entry under b_{1k} by subtracting the first row from the $k + 2$ row in both matrices. This transformation spoils the first column of the first matrix. We repair it by the following transformations: subtract the $2, \dots, m$ rows from the $k + 3, \dots, k + m + 1$ rows, respectively. Then add the $k + 1, \dots, k + m - 1$ columns to the

$1, \dots, m-1$ columns, respectively, and obtain

$$\left(\left[\begin{array}{c|c|c} \begin{matrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{matrix} & & \\ \hline & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \\ \hline & & \begin{matrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{matrix} \end{array} \right], \left[\begin{array}{c|c|c} \begin{matrix} * & \dots & * \\ 1 & & 0 \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 0 & \dots & 0 & 1 \end{matrix} & \begin{matrix} * & \dots & * \end{matrix} \\ \hline & \begin{matrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 & 0 \end{matrix} & \\ \hline \begin{matrix} * & \dots & * \end{matrix} & & \begin{matrix} * & \dots & * \\ 1 & & 0 \\ & \ddots & \\ & & 1 \end{matrix} \end{array} \right] \right). \quad (12)$$

Permute the first and second vertical strips, then the first and second horizontal strips, and get the pair

$$\left(\left[\begin{array}{c|c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & 1 \end{matrix} & & \\ \hline & \begin{matrix} 1 & & \\ 0 & 1 & \\ & 0 & \ddots \\ & & \ddots & 1 \\ & & & 0 \end{matrix} & \\ \hline & \begin{matrix} 0 & * & \dots & * \\ & 0 & \ddots & \vdots \\ & & \ddots & * \\ & & & 0 \end{matrix} & \begin{matrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{matrix} \end{array} \right], \left[\begin{array}{c|c|c} \begin{matrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 & 0 \end{matrix} & & \\ \hline & \begin{matrix} 1 & * & * & \dots & * \\ & 1 & & & 0 \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{matrix} & \begin{matrix} * & \dots & * \\ * & \dots & * \end{matrix} \\ \hline & \begin{matrix} * & * & \dots & * \\ & * & \ddots & \vdots \\ & & \ddots & * \\ & & & * \end{matrix} & \begin{matrix} * & \dots & * \\ 1 & & 0 \\ & \ddots & \\ & & 1 \end{matrix} \end{array} \right] \right), \quad (13)$$

in which we replace by stars some zero entries of the $(3, 2)$ blocks.

Denote by A_{ij} and B_{ij} ($i, j = 1, 2, 3$) the blocks in the first and second matrices of (13). By transformations from Lemma 1, we make zero all stars in B_{33} ; the form of other blocks does not change. Make zero the first row of B_{32} by adding $2, 3, \dots$ rows of the second horizontal strip to the first row of the third strip simultaneously in both matrices. Make zero the first row of A_{32} by adding the first column of the third vertical strip simultaneously in both matrices. Then, adding $3, 4, \dots$ rows of the second strip to the second row of the third strip, we make zero the second row of B_{32} . Adding the second column of the third vertical strip we make zero the second row of A_{32} , and so on until obtain (13) without stars in the third horizontal strip of both matrices.

Using Lemma 1, we make zero all stars in B_{22} . Multiplying the second horizontal strip in both matrices by an arbitrarily small number and then dividing the second vertical strips by the same number, we make the entries of B_{23} by arbitrarily small; these transformations do not change the other blocks. Denote the obtained pair by \mathcal{P}_1 ; it is an arbitrarily small deformation of $(F_{k+1}, G_{k+1}) \oplus (F_{l+1}, G_{l+1})$ in which k and l are defined in (11), and so

$$k + 1 + l + 1 = m + n, \quad k \geq m, \quad l \geq m.$$

Write $m_1 := \min(k + 1, l + 1)$ and $n_1 := \max(k + 1, l + 1)$.

Thus, the original pair (9) is equivalent to \mathcal{P}_1 being an arbitrarily small deformation of

$$(F_{m_1}, G_{m_1}) \oplus (F_{n_1}, G_{n_1}), \quad m < m_1 \leq n_1. \quad (14)$$

If \mathcal{P}_1 is a *nonzero* deformation of (14), then $m_1 < n_1$ because of the form of the miniversal deformation. Repeating the reasoning, we reduce \mathcal{P}_1 by equivalence transformations to a pair \mathcal{P}_2 being an arbitrarily small deformation of

$$(F_{m_2}, G_{m_2}) \oplus (F_{n_2}, G_{n_2}), \quad m < m_1 < m_2 \leq n_2.$$

If this deformation is nonzero, then we repeat the transformation, and so on until obtain \mathcal{P}_t being the *zero* deformation of

$$(F_{m_t}, G_{m_t}) \oplus (F_{n_t}, G_{n_t}), \quad m < m_1 < \dots < m_t \leq n_t.$$

We have proved that (9) is equivalent to $\mathcal{P}_t = (F_{m_t}, G_{m_t}) \oplus (F_{n_t}, G_{n_t})$.

\Leftarrow Let us suppose that we have a pair (A, B) of the form (3). Under arbitrary small deformations such a pair can't change to a direct sum of summands of transposed matrices: if pair has the form $(F_m, G_m) \oplus (F_{m+1}, G_{m+1})$, then summands of the type $((F_m^T, G_m^T) \oplus (F_{m+1}^T, G_{m+1}^T))$ can't be obtained. Moreover, since matrix is singular by rows and columns, its canonical form can't consist summands of types $(J_r(0), I_r)$ and $(I_r, J_r(0))$ and since we can always move from (A, B) to $(A, B - \lambda A^T)$, then the case of $(J_r(\lambda), I_r)$ can't be applied.

Thus, the canonical form of pair (A, B) of form form (3) under arbitrary small deformations can have only the form $((F_{m'}, G_{m'}) \oplus (F_{n'}, G_{n'}))$. Now we need to show that $m' = m + 1, n' = n - 1$. Moreover, we need to show that for each pair such a deformation exists.

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